On Back-reaction in Special Relativity

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Abstract

Back-reaction of fields plays an important role in the generation of particle masses and the mass-energy equivalence of special relativity, but the most natural demonstrations using classical models result in apparent errors such as the notorious "4/3" problem. Here we discuss the resolution of these discrepancies within the underlying atomic description of matter, with the aim of encouraging classroom discussion of back-reaction and its connection to relativistic effects.

I. INTRODUCTION

Equivalence between mass and energy is a cornerstone of special relativity, yet the formal derivations provide little physical intuition. Students can calculate quantitatively correct values for the masses of different configurations, e.g. a capacitor with varying plate separation, but often gain little or no qualitative understanding of the mechanisms which produce the differences.

One of the primary mechanisms has been understood, at least in principle, since the 1881 observation of J.J. Thomson that a charged, accelerating sphere experiences a "back-reaction" force from its own electromagnetic (EM) field, which acts to resist the acceleration and effectively increases the mass of the sphere.[1–3] This effect, and its extension to other systems of charges such as capacitors or atomic bound states, provides the mechanism by which internal EM energy contributes to the overall mass of a system. The magnetic component of the EM back-reaction should already be familiar to most students as the phenomenon of self-induction.

Different charge configurations lead to different back-reaction forces, which, along with internal kinetic energy such as electron orbital motion, accounts for most of the mass differential between different states of ordinary matter. Moreover, as has been emphasized by Wilczek,[4] back-reaction of the strong nuclear force accounts for most of the mass of protons and neutrons, only a small fraction of which comes from the intrinsic (or Higgs-derived) quark masses. Students who have studied self-induction therefore understand, in essence, one of the primary sources of mass in the universe, as well as one of the principal mechanisms of mass/energy equivalence, but they may not be aware of these connections.

Given the centrality of this mechanism it would be beneficial to expose students to it, for example by computing back-reaction in simple classical configurations and showing its effect on mass. However, a vexing problem arises: computations of this sort do not give correct answers. One expects to find that the mass contributed by back-reaction equals the stored EM energy divided by c^2, as predicted by Einstein's formula, but model computations give answers differing from this by geometry-dependent factors, most notoriously the factor 4/3 for spherically symmetric cases; indeed, the problem has come to be known as the "4/3 problem".[5]

An explanation for such discrepancies was proposed long ago by Poincaré, who noted that

the charge distributions appearing in classical models are unstable and require additional forces to stabilize them.[6] Adding in the energy and momentum contributions from these "Poincaré stresses" compensates the discrepancy and allows the overall momentum and energy to transform properly as a 4-vector.[5]

Poincaré stresses do resolve the problem, but when treated generically they leave the mechanism rather obscure, and the focus on additional forces is confusing since everyday matter is, in fact, constructed from EM forces and charged particles. Here we clarify how the problem is resolved within realistic atomic matter, with the aim of encouraging classroom discussion of back-reaction examples. The 4/3 problem has, of course, been discussed many times previously, but most often in the context of constructing classical models of the electron, which is not our present interest.[7]

We view this as part of a broader effort to introduce more qualitative or constructive explanations into the teaching of special relativity, with the goal of improving students' intuition for relativistic effects and showing how those effects connect to prior elements of the physics curriculum.[8–11] No originality is claimed for the calculations to follow, and references have been cited when known. Natural units $c = \hbar = 1$ will be used throughout, and "mostly-plus" metric signature $(-,+,+,+)$.[12] Greek indices $\mu, \nu, \ldots$ refer to 4-space coordinates, while $i, j, \ldots$ refer to 3-space, with repeated indices summed in both cases.

II. MODELS OF BACK-REACTION AND RESULTING DISCREPANCIES

Perhaps the simplest configuration in which to study EM back-reaction is the parallel-plate capacitor, Fig.1. This also provides a model for bound states in general, since varying the separation of the plates changes the internal EM energy just as, e.g., for different atomic levels.

We consider a capacitor having square plates with sides of length L parallel to the x, y axes, separated by distance h in the z direction. The top plate holds uniform surface charge density $\sigma > 0$ and the bottom plate $-\sigma$, leading to the electric field $\vec{E} = -\sigma\hat{z}$.

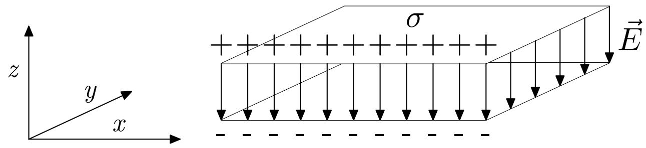

FIG. 1. Parallel-plate capacitor

The question to be answered is how much of the mass of the capacitor is due to its EM field. According to Einstein's formula the answer should be

$$\Delta M = \mathcal{H}_{EM} \tag{1}$$

where $\mathcal{H}_{EM}$ is the EM field energy, given by

$$\mathcal{H}_{EM} = \frac{1}{2}\int d^3x\, E^2. \tag{2}$$

One measures mass by trying to accelerate the system, i.e. by applying a force and measuring the total impulse required for the system to attain a given speed. Assuming that the external force doesn't interact with the internal EM field directly, the only way in which the internal EM field affects this measurement is through back-reaction onto the charges.

We consider first accelerating the capacitor in the $\hat{x}$ direction, in which case the back-reaction takes the form of self-induction. As the charges accelerate, a magnetic field $\vec{B}$ develops between the plates, parallel to $\hat{y}$. The growing B-field in turn creates an E-field which acts back on the charges, opposing the acceleration. Instead of following this process explicitly, we will calculate the total back-reaction impulse in a more general way which applies more easily to other configurations.

The back-reaction is just force applied by the EM field, i.e., momentum exchanged between the field and the matter. The total back-reaction impulse generated by an EM field between two time points is therefore the negative of the change in the EM field momentum. The EM field momentum is given by the Poynting vector[13]

$$\vec{\mathcal{P}}_{EM} = \int d^3x\, (\vec{E}\times\vec{B}) \tag{3}$$

and we note that the capacitor has $\vec{\mathcal{P}}_{EM} = 0$ when at rest. The cumulative back-reaction impulse from accelerating the capacitor to velocity $\vec{\beta}$ is then found by calculating $\vec{\mathcal{P}}_{EM}$ for the moving system, a calculation which is often quite easy because one can use a Lorentz transformation to find the EM field of the moving system.

Here, and in the remainder of the paper, we consider only small velocities, and work to first order in $\vec{\beta}$. First-order calculations suffice to determine the effective mass and exhibit the primary back-reaction mechanisms, while simplifying the calculations greatly.

The expected EM momentum in light of Einstein's formula (Eq.1) would be

$$\vec{\mathcal{P}}_{EM} = \vec{\beta}\mathcal{H}_{EM}, \tag{4}$$

but the actual situation is not so simple. Applying the standard Lorentz transformation of the EM field, the boosted state acquires a magnetic field $\vec{B} = \vec{\beta} \times \vec{E}$, leading to momentum[2]

$$\begin{aligned} \vec{\mathcal{P}}_{EM} &= \int d^3x \, (\vec{E} \times (\vec{\beta} \times \vec{E})) \\ &= \int d^3x \, (\vec{\beta} E^2 - \vec{E}(\vec{\beta} \cdot \vec{E})), \end{aligned} \tag{5}$$

a formula which applies to any configuration having negligible initial B-field.

For the capacitor accelerated along $\hat{x}$, the second term vanishes, and by comparing with Eqs.(2,4) one sees immediately that the result is too large by a factor of two! Hence, although the example clearly shows a qualitative effect of back-reaction on mass, it does not successfully reproduce the quantitative mass/energy relation.

For acceleration in the $\hat{z}$ direction the situation is equally perplexing, for no B-field is generated and the final EM momentum is also zero.[14] It would appear that there is no back-reaction at all, hence no contribution from the EM energy to the mass of the system. One might say that the $\hat{x}$ case has a "factor of two" discrepancy, while the $\hat{z}$ case has a "factor of zero" discrepancy. In fact the discrepancy was discovered in studying spherically-symmetric geometries (models of the electron) and has long been referred to as the "4/3 problem", after the factor which is found for these cases. We note that, in addition to self-induction, one must usually also include electrostatic self-forces; however, Eq.(5) remains valid.

If the momentum stored in the EM field does not account for the relativistically-required contribution Eq.(4), the difference must be stored somewhere else, since the underlying theory of quantum electrodynamics does respect both momentum conservation and Lorentz invariance. The only place additional momentum could be stored is in the surrounding atomic material of the capacitor, hence there must be a compensating momentum discrepancy in the atomic matter; in other words, the total momentum of the electrons and nuclei in the moving atomic system must also differ from that expected by simply applying Einstein's

formula (the analog of Eq.4 for the matter subsystem). In the following sections we show how this "matter momentum discrepancy" arises.

For future reference we show the general form of the EM momentum discrepancy, obtained by subtracting Eq.(4) from Eq.(5):

$$\Delta\vec{\mathcal{P}}_{EM} = \int d^3x\, (\frac{1}{2}\vec{\beta}E^2 - \vec{E}(\vec{\beta}\cdot\vec{E})). \tag{6}$$

III. MOMENTUM DISCREPANCY IN THE MATTER SYSTEM

If matter consisted of a static assembly of components then it would be hard to see how an object could harbor any momentum beyond that of its bulk motion; however, the modern view of matter involves a great deal of internal motion which can, in fact, develop a net momentum in relativistic theories. We consider first a classical atomic model consisting of a uniformly distributed ring of identical charged particles in circular orbits at fixed radius around a central charged nucleus. We neglect radiation, and assume that the total orbiting charge density is small enough that the charges have negligible effect on each other's motion. The force on a charge q located at position $\vec{r}$ relative to the nucleus is $\vec{F} = -q|E|\hat{r}$, where $|E|$ is constant.

When the system (i.e., the nucleus) is at rest, the orbital momenta of the charges cancel and add nothing to the total momentum. In Newtonian physics this would also be true of the moving system, but in a relativistic theory it need not be. We consider boosting the system to a small velocity $\vec{\beta}$, where $\vec{\beta}$ lies in the plane of the orbits (Fig.2). The orbits remain circular to first order, since length contraction starts at $\mathcal{O}(\beta^2)$, but they receive two types of first-order correction.

First, the orbital velocities are altered due to the relativistic addition-of-velocity formula. This change does not, however, lead to any momentum discrepancy, since it corresponds to taking a snapshot of the particle momenta at a fixed time and then Lorentz-transforming them all. The resulting total 4-momentum is just the Lorentz transform of the original total 4-momentum, with no extra discrepancy; in other words, this part gives rise to the analog of Eq.(4) for the orbiting particles.

Transforming all the momenta in a fixed-time snapshot does not, however, give an accurate picture of the moving system, since the definition of fixed time differs between frames. Because the orbiting particles are continually accelerating, this difference can translate into

an extra contribution to the total momentum. More concretely, the varying orbital velocities of the particles also imply a varying density, which creates net momentum.

It is easiest to calculate the momentum discrepancy directly in terms of the internal force $\vec{F}(\vec{r})$ and the relativistic simultaneity change. The boosted trajectories are obtained, to first order, by substituting the coordinate change

$$\begin{aligned} \vec{x} \rightarrow \vec{x}' &= \vec{x} - \vec{\beta} t \\ t \rightarrow t' &= t - \vec{\beta} \cdot \vec{x} \end{aligned} \tag{7}$$

into the rest-frame trajectories. The momentum discrepancy arises from the simultaneity shift,

$$\delta t = -\vec{\beta} \cdot \vec{x}, \tag{8}$$

which effectively shifts the particles forwards or backwards in their trajectories by δt, giving rise to the density shifts shown schematically in Fig.2. During time δt, a particle at location $\vec{r}$ changes its momentum by $\vec{F}(\vec{r})\delta t$, hence the overall momentum discrepancy is

$$\begin{aligned} \delta \vec{P} &= - \sum_{charges} (\vec{\beta} \cdot \vec{x}) \vec{F} \\ &= - \int d^3x \, (\vec{\beta} \cdot \vec{x}) \rho \vec{E}. \end{aligned} \tag{9}$$

In the last line, the equation has been written in more general form as an integral against the orbital charge distribution ρ, anticipating calculations to follow.

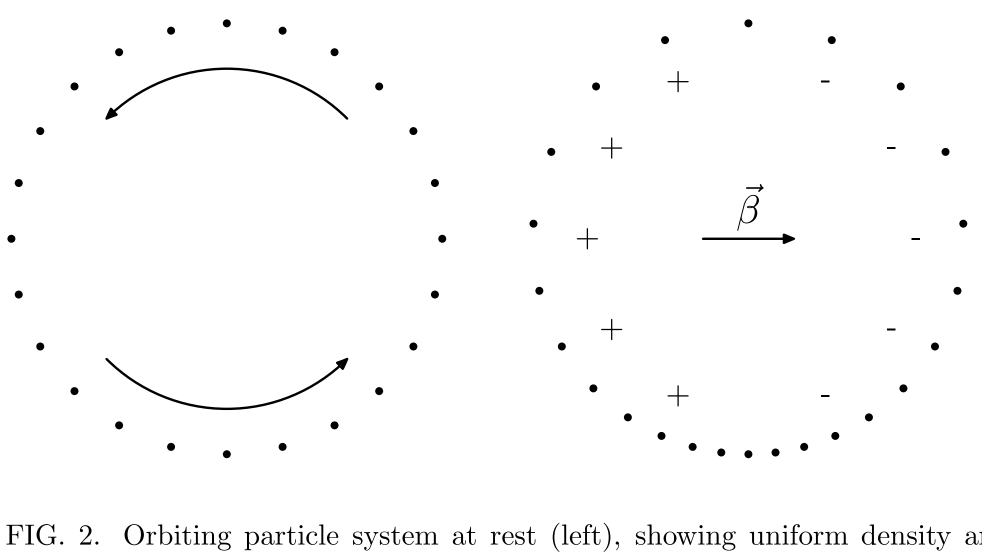

FIG. 2. Orbiting particle system at rest (left), showing uniform density around the orbit, and boosted (right), showing varying density. Arrows at left indicate the orbital velocity. Symbols $+, -$ at right indicate the sign of δt, Eq.8.

For the circular orbits of the model, $\delta\vec{P}$ is nonzero since both $\vec{\beta}\cdot\vec{x}$ and $\vec{E}$ are odd under reflection through the origin; hence, the orbital momentum develops a discrepancy, and the next step is to compare this to the EM momentum discrepancy, Eq.(6). For this we first need to rewrite Eq.(9) to include the full charge distribution, i.e. the nuclear charge as well, since Eq.(6) includes the E-fields arising from all charges. We define ρ_1, ρ_2 to be the orbital and nuclear charge densities, respectively, and $\vec{E}_1, \vec{E}_2$ to be the respective electric fields resulting from these charges. We can then write

$$\delta\vec{P} = -\int d^3x\,(\vec{\beta}\cdot\vec{x})(\rho_1\vec{E}_2 + \rho_2\vec{E}_1) \tag{10}$$

where the first term is just Eq.(9), while the second term vanishes since $\vec{E}_1$ is zero at the center of the orbit, where ρ_2 is concentrated.

Since the B-field is negligible in the system at rest, the fields $\vec{E}_i$ are derived from scalar potentials, $\vec{E}_i = -\vec{\nabla}\phi_i$, and one has also $\nabla^2\phi_i = -\rho_i$. Inserting the scalar potentials and integrating by parts several times, one arrives at

$$\delta\vec{P} = \int d^3x\,\Big(-\vec{\beta}(\vec{E}_1\cdot\vec{E}_2) + \vec{E}_1(\vec{\beta}\cdot\vec{E}_2) + \vec{E}_2(\vec{\beta}\cdot\vec{E}_1)\Big)\,. \tag{11}$$

(This result is also shown through a slightly more general method in Sec.V.)

This may now be compared to the EM momentum discrepancy, Eq.(6), with $\vec{E} = \vec{E}_1 + \vec{E}_2$, and one sees that the matter discrepancy cancels the "interaction" portion of the EM momentum discrepancy, i.e., the terms containing both $\vec{E}_1$ and $\vec{E}_2$.[2] It does not cancel the "self-energy" terms which are quadratic in $\vec{E}_1$ or $\vec{E}_2$ alone; this makes sense because the self-energies are already implicitly included in the particle masses of the classical model. We note also that it is the interaction energy which is relevant for understanding mass differentials between different configurations of the same charges, e.g. a capacitor with varying plate separation.

Eq.(11) and its comparison to the EM momentum discrepancy are, intuitively, the main results, but the classical model is too far removed from the true atomic theory to be fully convincing. In addition to many other well-known deficiencies, one may doubt that the orbiting system would, in fact, attain the same state through physical acceleration which is found by Lorentz transformation, because the orbits are largely unconstrained and could change in many ways. This doubt is considerably reduced in the wave theory of atomic orbitals due to the discreteness of the solutions, since a slow acceleration will not typically

cause a jump between different solutions.[15] For these reasons, we recalculate the momentum discrepancy in the following section using the Dirac formalism.

IV. MOMENTUM DISCREPANCY IN THE DIRAC THEORY

For a Dirac particle of charge q coupled to an EM field $(\phi, \vec{A})$, the Hamiltonian and momentum operators are (see Appendix and ref. 16)

$$\begin{aligned} \hat{H} &= \hat{\alpha}^i \hat{p}_i + \hat{\beta} m + q\phi \\ \hat{p}^i &= -i(\partial^i - iqA^i), \end{aligned} \tag{12}$$

where $\hat{\alpha}^i$, $\hat{\beta}$ are the standard matrices of Dirac ($\hat{\beta}$ is unrelated to the boost velocity β^i). We will consider just one Dirac field, and work in the single-particle formalism, ignoring the complications of multi-electron states and the additional field for the nucleus; these do not fundamentally change the calculation.

Given a wavefunction ψ which satisfies the Dirac equation, boosting the wavefunction means substituting the coordinate changes, Eq.(7), and applying a spinor rotation given by $\frac{1}{2}(\vec{\beta} \cdot \vec{\hat{\alpha}})$.[16] Working for convenience at time $t = 0$, the only coordinate change is $\delta t = -\vec{\beta} \cdot \vec{x}$, the effect of which can be computed using the Hamiltonian:

$$\begin{aligned} \delta\psi &= -(\vec{\beta} \cdot \vec{x})\partial_0 \psi + \frac{1}{2}(\vec{\beta} \cdot \vec{\hat{\alpha}})\psi \\ &= i(\vec{\beta} \cdot \vec{x})\hat{H}\psi + \frac{1}{2}(\vec{\beta} \cdot \vec{\hat{\alpha}})\psi. \end{aligned} \tag{13}$$

The change in expected momentum is then given by

$$\delta \left\langle P^i \right\rangle = \delta \left(\int d^3x \, \psi^\dagger \hat{p}^i \psi \right) = \int d^3x \, \left(\delta\psi^\dagger \hat{p}^i \psi + \psi^\dagger \hat{p}^i \delta\psi + \psi^\dagger \delta\hat{p}\psi \right). \tag{14}$$

We choose a gauge where $\vec{A} = 0$ in the rest frame (again assuming negligible $\vec{B}$ field), so the boosted state has $\vec{A} = \vec{\beta}\phi$ (an additional gauge transformation could accompany the boost, but there is no reason to consider this). The change in $\vec{A}$ implies $\delta\hat{p}^i = -q\phi\beta^i$, and we will

also need to use $[\hat{H}, (\vec{\beta}\cdot\vec{x})] = -i\vec{\beta}\cdot\vec{\alpha}$, which cancels the spinor rotation terms. One finds

$$\begin{aligned}
\delta\left\langle P^i\right\rangle &= \int d^3x\ \psi^\dagger\left(\left[-i\hat{H}(\vec{\beta}\cdot\vec{x}) + \frac{1}{2}(\vec{\beta}\cdot\vec{\alpha})\right]\hat{p}^i + \hat{p}^i\left[i(\vec{\beta}\cdot\vec{x})\hat{H} + \frac{1}{2}(\vec{\beta}\cdot\vec{\alpha})\right] - q\phi\beta^i\right)\psi \\
&= \int d^3x\ \left(-i(\vec{\beta}\cdot\vec{x})\psi^\dagger[\hat{H},\hat{p}^i]\psi + i\psi^\dagger[\hat{p}^i,\vec{\beta}\cdot\vec{x}]\hat{H}\psi - q\phi\beta^i\psi^\dagger\psi\right) \\
&= \beta^i\left\langle\hat{H} - q\phi\right\rangle + \int d^3x\, q(\vec{\beta}\cdot\vec{x})\partial^i\phi\psi^\dagger\psi \\
&= \beta^i\left\langle\hat{H} - q\phi\right\rangle - \int d^3x\,(\vec{\beta}\cdot\vec{x})\left\langle\rho\right\rangle E^i. \qquad (15)
\end{aligned}$$

The first term is the expected (non-discrepancy) term, and contains the Dirac single-particle energy minus the EM interaction energy; this is the correct energy to use in conjunction with the EM expression Eq.(2), because the latter already includes all matter interaction energies (see Appendix). The second term is the momentum discrepancy, confirming the result found from the classical model, Eq.(9).

This calculation, based as it is on a fixed background EM field $(\phi, \vec{A})$, still does not account for self-energies, which are again implicitly included in the particle masses. A full calculation would require quantum field theory and renormalization, but otherwise would follow the same manipulations leading to Eq.(15), and indeed would lead to the same result, at lowest order.

V. THE EM MOMENTUM DISCREPANCY MORE GENERALLY

The formula (Eq.6) for the EM momentum discrepancy involved only the EM field, while the matter momentum discrepancy (e.g., Eq.9) involved both the EM field and the charge density. The computation outlined above Eq.(11) shows that the two forms are, in fact, opposite to each other, but it is worthwhile to show this using a slightly more general method.

The EM energy and momentum derive from the stress-energy tensor (see Appendix)

$$T_\mu{}^\nu = -F_{\mu\alpha}F^{\nu\alpha} + \frac{1}{4}\delta_\mu{}^\nu F^{\alpha\beta}F_{\alpha\beta} \qquad (16)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the EM field strength. In the presence of charges, $T_\mu{}^\nu$ is not conserved but rather satisfies

$$\partial^\mu T_\mu{}^\nu = F^{\nu\alpha}J_\alpha \qquad (17)$$

where J^μ is the total charge current, with $\rho = J^0$. The EM momentum and energy Eqs.(2,3) are given by

$$\mathcal{P}^i_{EM} = \int d^3x\, T_0{}^i,$$
$$\mathcal{H}_{EM} = \int d^3x\, T_0{}^0 \tag{18}$$

and we wish to calculate the variation of $\mathcal{P}^i_{EM}$ under the small boost Eq.(7). This is given by the tensor index transformations plus the time change:

$$\delta\mathcal{P}^i_{EM} = \int d^3x\, \{\beta^i T_0{}^0 - \beta^j T_j{}^i - (\vec{\beta}\cdot\vec{x})\partial_0 T_0{}^i\}. \tag{19}$$

In the last term we can substitute Eq.(17) and integrate the resulting spatial derivatives by parts, cancelling the second term and arriving at

$$\delta\mathcal{P}^i_{EM} = \beta^i\mathcal{H}_{EM} + \int d^3x\, (\vec{\beta}\cdot\vec{x})F^{i\alpha}J_\alpha \tag{20}$$

which, for negligible $\vec{B}$ field, and noting $E^i = F^{0i}$, reduces to

$$\delta\mathcal{P}^i_{EM} = \beta^i\mathcal{H}_{EM} + \int d^3x\, (\vec{\beta}\cdot\vec{x})\rho E^i \tag{21}$$

showing a discrepancy exactly opposite to the matter momentum discrepancy of Eqs.(9,15).

VI. CONCLUSION

The apparent discrepancies in back-reaction calculations involving classical charge distributions are thus accounted for by compensating discrepancies within the materials holding the charges. This is precisely the mechanism proposed by Poincaré many years ago; our purpose has been to describe it more concretely in terms of the underlying matter theories which are now known to hold.

More specifically, the external electric field which is sourced by the classical charges (e.g., charges on the capacitor plates) develops a momentum discrepancy Eq.(6), which seems puzzling because the classical charges are fixed in place and cannot store additional momentum. However, Eq.(11) shows that the full EM momentum discrepancy, both external and internal, is cancelled by the corresponding matter discrepancies, when all charges are included. The classical charges do not contribute to this because (noting Eq.9) the total E-field at their locations must vanish in order for the charges to be static.

Discrepancies of the "4/3" variety are seen to be inevitable when boosting a composite relativistic system, because the boost alters the simultaneity surface differently in different directions, changing the way momentum is divided between the subsystems. Another way to understand the discrepancies is to note that the back-reaction from one subsystem onto another arises from a local interaction, which has no way of accounting for the total energy contained in the extended region occupied by the subsystem. This was not an issue in Newtonian physics because interactions in that paradigm are not mediated by local fields, hence do not cause energy and momentum to be dispersed over an extended spatial region to begin with.

A more formal way to see the orgin of discrepancies is to observe that the interaction Lagrangian $\mathcal{L}_I$ typically contains no time derivatives (cf. Appendix), hence does not contribute to the momentum density ${T_0}^i$, but the energy ${T_0}^0$ does contain $-\mathcal{L}_I$. The boosted momentum must then contain a term $-\vec{\beta}\mathcal{L}_I$, and this can only be produced through discrepancies in the subsystem momenta.

APPENDIX: DERIVATION OF ENERGY-MOMENTUM EXPRESSIONS

For completeness in understanding Sections IV and V, we show the derivation of fundamental energy- and momentum-related quantities for the EM and Dirac fields. The action for an EM field coupled to fermions of charge q is given by $\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_\psi$, where[12]

$$\begin{aligned} \mathcal{L}_{EM} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ \mathcal{L}_\psi &= -\bar{\psi}\gamma^\mu(\partial_\mu - iqA_\mu)\psi - m\bar{\psi}\psi \end{aligned} \tag{22}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The EM field satisfies the equation

$$\partial_\mu F^{\mu\nu} = -J^\nu \tag{23}$$

where J^ν is the charge current $iq\bar{\psi}\gamma^\nu\psi$. We note that in the mostly-plus metric convention one has $\bar{\psi} = i\psi^\dagger\gamma^0$, giving $J^0 = -q\psi^\dagger(\gamma^0)^2\psi = q\psi^\dagger\psi$, which is the expected form of the charge density.

The matter equation of motion is

$$\gamma^\mu(\partial_\mu - iqA_\mu)\psi + m\psi = 0 \tag{24}$$

from which one derives

$$\begin{aligned}\partial_0\psi &= \gamma^0\gamma^i(\partial_i - iqA_i)\psi + m\gamma^0\psi + iqA^0\psi \\ &= -i\hat{H}\psi \end{aligned} \tag{25}$$

where the single-particle Dirac Hamiltonian $\hat{H}$ is as in Eq.(12), and we note the mostly-plus definitions $\hat{\beta} = i\gamma^0$ and $\hat{\alpha}^i = -\gamma^0\gamma^i$, as well as $\phi = A^0$.

The canonical stress-energy (SE) tensors are given by[13]

$$\begin{aligned}(T_{EM})^\mu{}_\nu &= \frac{\delta\mathcal{L}_{EM}}{\delta\partial_\mu A^\alpha}\partial_\nu A^\alpha - \delta^\mu{}_\nu\mathcal{L}_{EM} \\ &= -F^{\mu\alpha}\partial_\nu A_\alpha - \delta^\mu{}_\nu\mathcal{L}_{EM}\end{aligned} \tag{26}$$

and

$$\begin{aligned}(T_\psi)^\mu{}_\nu &= \frac{\delta\mathcal{L}_\psi}{\delta\partial_\mu\psi^\sigma}\partial_\nu\psi^\sigma - \delta^\mu{}_\nu\mathcal{L}_\psi \\ &= -\bar{\psi}\gamma^\mu\partial_\nu\psi - \delta^\mu{}_\nu\mathcal{L}_\psi.\end{aligned} \tag{27}$$

The total energy and momentum of the system are then defined by

$$\begin{aligned}\mathcal{H} &= \int d^3x\ \left\{(T_{EM})_0{}^0 + (T_\psi)_0{}^0\right\}, \\ \mathcal{P}^i &= \int d^3x\ \left\{(T_{EM})_0{}^i + (T_\psi)_0{}^i\right\}.\end{aligned} \tag{28}$$

These overall quantities are gauge invariant since they are derived from a gauge-invariant Lagrangian, but the separate EM and matter tensors are not gauge invariant, and using $(T_{EM})_\mu{}^\nu$ one does not obtain the familiar expressions Eqs.(2,3). To avoid this inconvenience one usually redefines the EM tensor to the gauge-invariant form

$$(\bar{T}_{EM})_\mu{}^\nu = -F_{\mu\alpha}F^{\nu\alpha} - \delta_\mu{}^\nu\mathcal{L}_{EM}, \tag{29}$$

which gives the normal expressions Eqs.(2,3). This change corresponds to adding an extra term

$$\Delta T_\mu{}^\nu = F_\mu{}^\alpha\partial_\alpha A^\nu \tag{30}$$

to the canonical EM tensor, and the same must then be subtracted from the matter tensor. Both SE tensors are then separately gauge invariant. We note that the overall SE tensor

is not symmetric, although the EM side becomes symmetric after the correction. The fully symmetrized SE tensor is obtained by adding a further total divergence to the matter SE tensor, which does not change the integrated energy and momentum.[13]

Hence, to verify that matter momentum discrepancies cancel those generated by the standard EM expressions Eqs.(2,3), one must use energy and momentum derived from the matter SE tensor

$$(\bar{T}_\psi)_\mu{}^\nu = (T_\psi)_\mu{}^\nu - F_\mu{}^\alpha \partial_\alpha A^\nu \tag{31}$$

which looks strange, but gives a reasonable result because the added term can be integrated by parts and then converted to a ψ term using Eq.(23). In this way one finds energy

$$\begin{aligned}\mathcal{H}_\psi &= \int d^3x \ \{(T_\psi)_0{}^0 - F_0{}^\alpha \partial_\alpha A^0\} \\ &= \int d^3x \ \{(T_\psi)_0{}^0 + J_0 A^0\} \\ &= \int d^3x \ \psi^\dagger(\hat{H} - q\phi)\psi \end{aligned} \tag{32}$$

showing the correctness of the energy used in Eq.(15). For the Dirac field momentum one finds similarly that the added SE term converts derivatives ∂_i to the gauge-invariant form $\partial_i - iqA_i$, justifying the gauged momentum operator given in Eq.(12). We note lastly that the SE tensor alteration Eq.(30) does not affect the momentum discrepancies in the two subsystems, as this term by itself has no discrepancy.

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